Betti numbers of graphs

Quantitative Methods in Defense and National Security

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25 May 2010

Overview

- Nexus of combinatorics and commutative algebra
- To each graph (combinatorial object), can uniquely associate an ideal (algebraic object)
- Understanding the ideal structure sheds light on the combinatorial object
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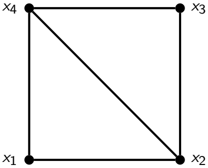
- (If you're not paying attention, some combinatorial topology creeps in as well).
- Largely due to Stanley, Villarreal, Katzman, Jacques, Hà, Van Tuyl

Graphs

- A *graph*, G, is a collection of vertices, $V = \{x_1, \ldots, x_n\}$, along with a set of edges, $E \subset V \times V$
- We are only considering undirected graphs, so write $\{x_i, x_j\}$ for the edge joining vertices x_i and x_j.
- A graph is *simple* if no vertex is joined to itself by an edge
- We allow at most one edge between any pair of vertices
- Obvious pictorial representation of the graph G with $V = \{x_1, \ldots, x_4\}$ and $E = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_1\}, \{x_2, x_4\}\}$:



Edge Ideal

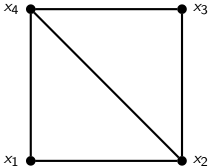
- Let G be a graph on n vertices, $x_1, \ldots, x_n$, and let R be the polynomial ring in n indeterminants; i.e., $R = \mathbb{C}[x_1, \ldots, x_n]$.
- We define the *edge ideal* of G, $I(G)$, to be the ideal "generated by the edges of G"; i.e.,

$$I =< x_i x_j \mid \{x_i, x_j\} \in E >$$

- A few remarks:
  - Edge ideal is quadratic, square-free monomial ideal
  - Bijective correspondence between such ideals and class of graphs we are considering
  - Isomorphic graphs have isomorphic edge ideals
  - Choice of coefficient field
  - Special case of Stanley-Reisner ideals of simplicial complexes

Illustration

- Recall earlier example of graph G with $V = \{x_1, \ldots, x_4\}$ and $E = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_1\}, \{x_2, x_4\}\}$:



- Then the edge ideal is:

$$I(G) = < x_1x_2, x_2x_3, x_3x_4, x_1x_4, x_2x_4 > \subset R (\equiv \mathbb{C}[x_1, \ldots, x_4])$$

Free Resolution

- **Henceforth**, consider I and R/I to be R-modules!
- Standard gadget from homological algebra is minimal free resolution of a module
- A *free resolution* of an R-module, M, is a complex of free modules

$$\mathfrak{F}: \qquad \ldots \xrightarrow{\phi_{n+1}} F_n \xrightarrow{\phi_n} \ldots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

  which is exact and is such that coker $\phi_1 = M$.
- Resolution is *graded* if R is a graded ring, the F_i are graded modules, and the maps are homogeneous of degree zero.
- A *finite free resolution of length n* is one in which $F_i = 0$ for all $i > n$ and $F_0, \ldots F_n$ are all nonzero

- Often we will write

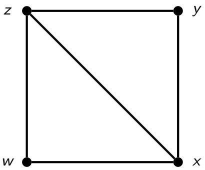
$$\mathfrak{F}: \qquad \ldots \xrightarrow{\phi_{n+1}} F_n \xrightarrow{\phi_n} \ldots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

and say that this is a resolution of M
- Easy to see that every finitely generated module has a free resolution:
  - Take a set of generators for M and map a free module, F_0, onto M in the obvious way
  - Let $M_1 \subset F_0$ be the kernel of this map and repeat

Minimal Free Resolutions

- *Hilbert Syzygy Theorem*: Let $R = \mathbb{C}[x_1, \ldots, x_n]$. Every finitely generated graded R-module has a graded finite free resolution, of length no more than n, by finitely generated free modules.
- A *minimal finite free resolution* of an R-module M is one with the smallest possible length and smallest possible rank for each of the free modules.
- Minimal resolutions exist and are unique, up to isomorphism
- The rank of the i^{th} free module in a minimal free resolution is called the i^{th} *Betti number* of M
- The *projective dimension* of a module is the length of its minimal resolution.
- Remark: Novel graph invariants — Betti numbers and projective dimension
- Remark: Betti sequence does not characterize the graph

Example 1



$$\leftrightarrow \quad I = < wx, xy, yz, wz, xz >$$

- Has minimal free resolution:

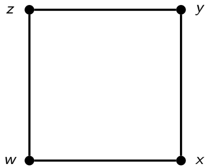
$$R^2 \xrightarrow{\begin{pmatrix} w & 0 \\ 0 & w \\ 0 & -y \\ 0 & -x \\ -y & 0 \\ z & z \end{pmatrix}} R^6 \xrightarrow{\begin{pmatrix} 0 & x & 0 & w & 0 & 0 \\ y & 0 & 0 & 0 & w & 0 \\ 0 & 0 & x & -y & 0 & 0 \\ -z & -z & 0 & 0 & 0 & w \\ 0 & 0 & -z & 0 & -z & -y \end{pmatrix}} R^5 \xrightarrow{\begin{pmatrix} yz & xz & wz & xy & wx \end{pmatrix}} R \xrightarrow{\pi} R/I \xrightarrow{0} 0$$

- Projective dimension is 3

  Betti numbers are:

  $\beta_0 = 1, \qquad \beta_1 = 5, \qquad \beta_2 = 6, \qquad \beta_3 = 2$

Example 2



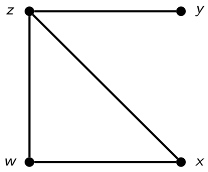
$$\leftrightarrow \quad I =< wx, xy, yz, wz >$$

- Has minimal free resolution:

$$\mathrm{R} \xrightarrow{\begin{pmatrix} w \\ -y \\ -x \\ z \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} x & 0 & w & 0 \\ 0 & x & -y & 0 \\ -z & 0 & 0 & w \\ 0 & -z & 0 & -y \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} yz & wz & xy & wx \end{pmatrix}} R \xrightarrow{\pi} R/I \xrightarrow{0} 0$$

- Projective dimension is 3
  Betti numbers are:
  $\beta_0 = 1, \qquad \beta_1 = 4, \qquad \beta_2 = 4, \qquad \beta_3 = 1$

Example 3



$$\leftrightarrow \quad I = < wx, yz, wz, xz >$$

- Has minimal free resolution:

$$R \xrightarrow{\begin{pmatrix} w \\ -y \\ -x \\ y \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} x & 0 & w & 0 \\ -y & 0 & 0 & w \\ 0 & x & -y & 0 \\ 0 & -z & 0 & -z \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} yz & xz & wz & wx \end{pmatrix}} R \xrightarrow{\pi} R/I \xrightarrow{0} 0$$

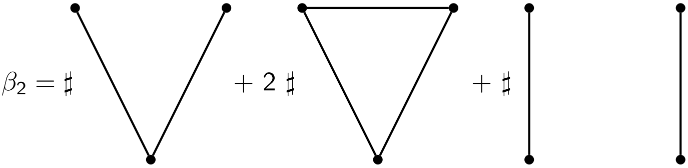
- Projective dimension is 3

  Betti numbers are:

  $\beta_0 = 1, \qquad \beta_1 = 4, \qquad \beta_2 = 4, \qquad \beta_3 = 1$

Betti Numbers and Graphs

- Hilbert syzygy theorem implies projective dimension is no greater than order of the graph
- β_0 is one
- β_1 is size
- Less obvious is



- Algebraic invariants capture graph combinatorics — Hochster's formula

Hochster's Formula

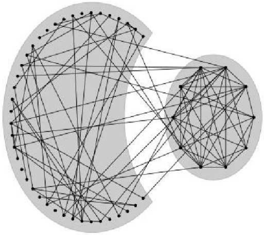
- (Hochster's Formula) The non-zero Betti numbers of G are only in square-free degrees $\mathbf{a}$ and may be expressed as

$$\beta_{i,\mathbf{a}}(G) = \dim \widetilde{H}_{|\mathbf{a}|-i-1}(\Delta_{\mathbf{a}})$$

- Hochster's formula thus relates the Betti numbers of a graph to structure of an associated (in some sense dual) simplicial complex.
- Primary tool in proving facts about Betti numbers of graphs (e.g., formulas for $\{\beta_i\}$ of various classes of graphs)

Hypothesis Testing

- Suppose that you've modeled a network as a graph; for example, communications between "bad guys"
  - Each "bad guy" is a vertex
  - If "a" talks to "b," join those vertices with an edge
- Question that could be interesting:
  - Are there regions of higher "density?"



Kidney-Egg

Hypothesis Testing

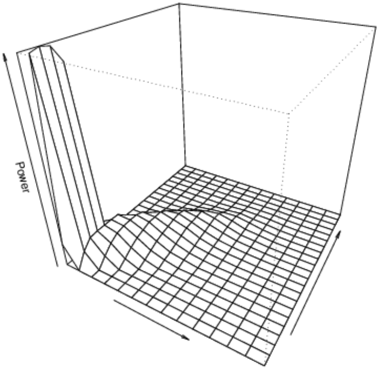
- Null hypothesis is Erdös-Rényi random graph
  Edges independent with probability p
- Alternative hypothesis is "kidney-egg"
  Region of higher edge probability, q
- Test to reject null hypothesis
  Size, average degree, Betti numbers, etc.
- "Power" of the test
  Probability that you correctly reject null hypothesis
- Hochster's formula relates Betti numbers to existence of various subgraphs, thus may have power to reject null
- Conversely, considering power in hypothesis testing may help gain insight into what the Betti numbers are capturing

"Real World" Considerations

- Exact computation of the minimal free resolution is expensive: combinatorial explosion
- Notion of "splitting edge" in a graph leads to recursive algorithm for computing $\{\beta_{ij}\}$ — Hà and Van Tuyl
- Can continue recursion iff graph is chordal
- Algorithms for approximating $\{\beta_{ij}\}$
  - "Pretend" an edge is splitting: hueristic for selecting non-splitting edges
  - Scan approach: compute $\{\beta_{ij}\}$ in neighborhood of each vertex and take maximum over the graph

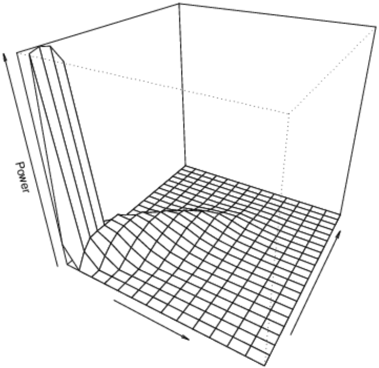
Power of $\{\beta_{ij}\}$

- Power of $\{\beta_{ij}\}$ to reject the null
  $(n = 50, p = .01, m = 8, q = .8,$ 1000 trials$)$



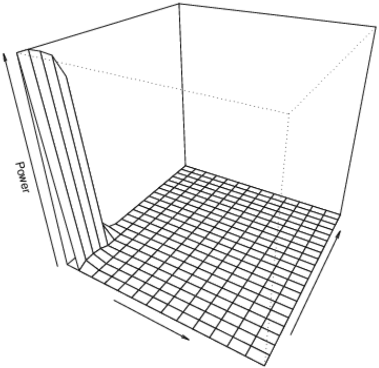
Power of approximate $\{\beta_{ij}\}$

- Power of approximate $\{\beta_{ij}\}$ to reject the null
  $(n = 50, p = .01, m = 8, q = .8,$ 1000 trials$)$



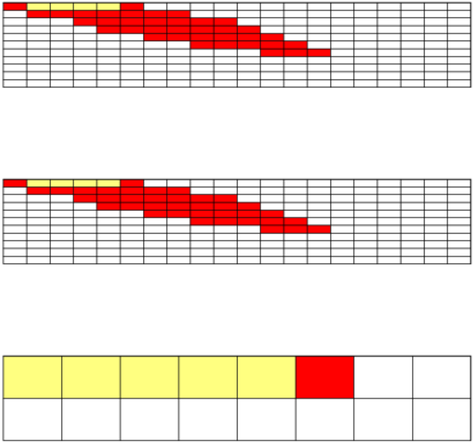
Power of scan $\{\beta_{ij}\}$

- Power of scan $\{\beta_{ij}\}$ to reject the null
  $(n = 50, p = .01, m = 8, q = .8,$ 1000 trials$)$



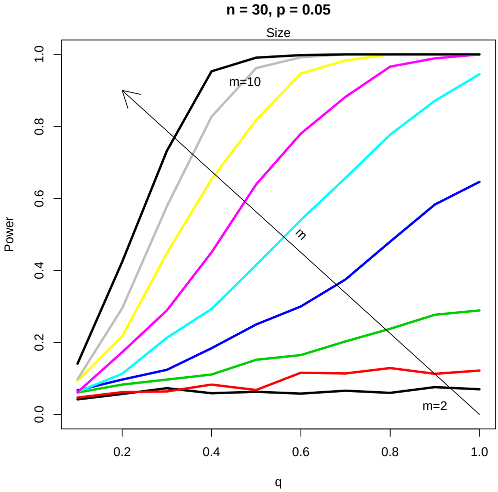
Comparative power of various $\{\beta_{ij}\}$

- Power of various $\{\beta_{ij}\}$ to reject the null
  (red = low power, yellow = high power)



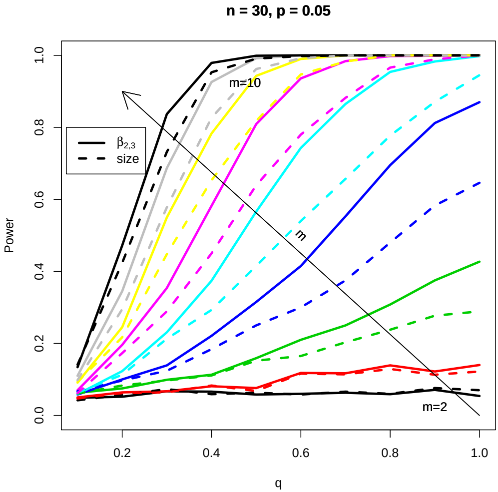
Results of Power

- Power of size to reject the null



Results of Power

- Comparison of power of size and $\beta_{2,3}$



Splitting Subgraphs

- Usual approach to computing Betti numbers lead to (inevitable?) combinatorial explosion
- Alternate approach:
  - "Split a graph into pieces"
  - Compute Betti numbers of each piece
  - Combine the component Betti numbers
- Algebraic condition is splittable monomial ideal (due to Eliahou and Kervaire)

  Expresses Betti numbers of entire ideal as sum of Betti numbers of certain subideals (and Betti numbers of an intersection term)

Splittable Ideal

- Let $\mathcal{G}(I)$ denote the minimal set of generators of a monomial ideal I
- A monomial ideal I is *splittable* if I is the sum of two nonzero monomial ideaLS J and K, that is, $I = J + K$, such that
  (1) $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$
  (2) there is a *splitting function*

$$\begin{aligned} \mathcal{G}(J \cap K) &\rightarrow \mathcal{G}(J) \times \mathcal{G}(K) \\ w &\mapsto (\phi(w), \psi(w)) \end{aligned}$$

  satisfying
    (a) for all $w \in \mathcal{G}(J \cap K)$, $w = \mathrm{lcm}(\phi(w), \psi(w))$
    (b) for every subset $S \subset \mathcal{G}(J \cap K)$, both $\mathrm{lcm}(\phi(S))$ and $\mathrm{lcm}(\psi(S))$ strictly divide $\mathrm{lcm}(S)$

If J and K satisfy the above properties, then we say that $I = J + K$ is a *splitting of* I.

Splittable Ideal

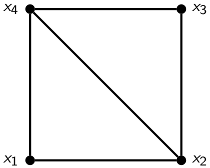
- Previously used $\mathbb{N}^n$ grading of $R = \mathbb{C}[x_1, \ldots, x_n]$; also an obvious $\mathbb{N}$ grading
- (Eliahou and Kervaire) Suppose that I is a splittable monomial ideal with splitting $I = J + K$. Then for all $i, j \geq 0$,

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)$$

- Note: Existence of a splitting function is sufficient for the Betti number formula to hold, but not a necessary condition.

Splitting Edge

- Work of Hà, Van Tuyl
- An edge $e = \{x_i, x_j\}$ is a splitting edge if $I(G) = (x_i x_j) + I(G \setminus e)$ is a splitting.
- Let G be a graph and $x_i \in V$ a vertex. The *(open) neighborhood* of $\{x_i\}$, $N(x_i)$ is the set of adjacent vertices; i.e., $N(x_i) = \{x_k \in V \mid \{x_i, x_k\} \in E\}$.



$$\Longrightarrow \quad \begin{aligned} N(x_1) &= \{x_2, x_4\} \\ N(x_2) &= \{x_1, x_3, x_4\} \end{aligned}$$

- (Hà, Van Tuyl) An edge $e = \{x_i, x_j\}$ is a splitting edge of G if and only if $N(x_i) \subset (N(x_j) \cup \{x_j\})$ or $N(x_j) \subset (N(x_i) \cup \{x_i\})$.
- $\{x_1, x_2\}$ above is a splitting edge.

Splitting Subgraphs

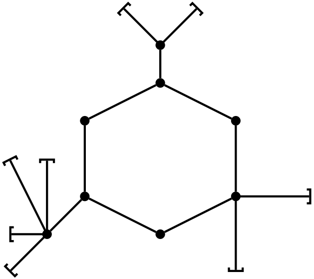
- Splitting edge work of Hà, Van Tuyl
  - Get a formula for Betti numbers in terms of subgraph: Theorem: Let $e = \{x_i, x_j\}$ be a splitting edge of G, and set $H = G \setminus (N(x_i) \cup N(x_j))$. If $n = |N(x_i) \cup N(x_j)| - 2$, then for all $i \geq 1$ and all $j \geq 0$

$$\beta_{i,j}(I(G)) = \beta_{i,j}(I(G \setminus e)) + \sum_{l=0}^{i} \binom{n}{l} \beta_{i-l-1,j-2-l}(I(H))$$

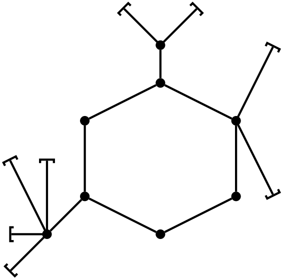
  - Leads to recursive algorithm ($\iff$ graph is chordal)
  - Induced cycles keep one from continuing recursion
- Considered "splitting cycle;" i.e., a cycle such that we can produce a splitting function.
  - If cycle is splitting, then can express Betti numbers of graph as sum of Betti numbers of cycle, its complement, and an intersection term
  - Proposition: A cycle is splitting if there are no adjacent vertices of degree greater than two.

Illustration

- Splitting is a condition on how the cycle connects to its complement (no adjacent vertices of degree greater than 2)
- Splitting cycle vs. not splitting



Splitting



Not Splitting

Summary

- Machinery allows one to produce novel graph invariants (and relate them back to the combinatorics of the graph)
- Bijective correspondence between simple, undirected graphs with no multiple edges and square-free, quadratic, monomial ideals allows one to illuminate the other.
- Only scratched the surface here today
  - Can bring much more algebraic sophistication to bear: fields of nonzero characteristic, regularity, Cohen-Macaulay, etc
  - Can bring much more combinatorial sophistication to bear: Stanley-Reisner rings, Taylor complexes, various notions of duality, etc