NAVAL SURFACE WARFARE CENTER
DAHLGREN DIVISION

ELECTROMAGNETIC & SENSOR SYSTEMS DEPARTMENT

Numerical Calculation of Particle Beam Limiting Currents

Quantitative Methods in Defense and National Security: 2010

Directed Energy Branch (Q22)

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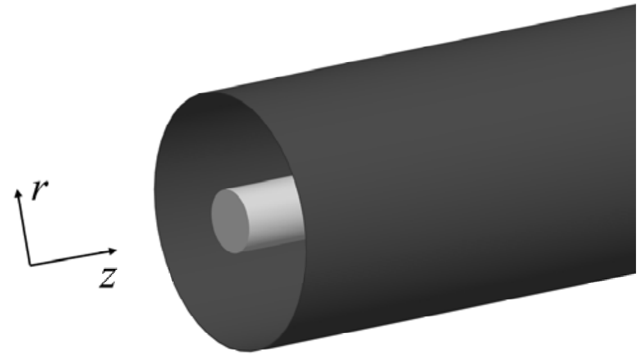
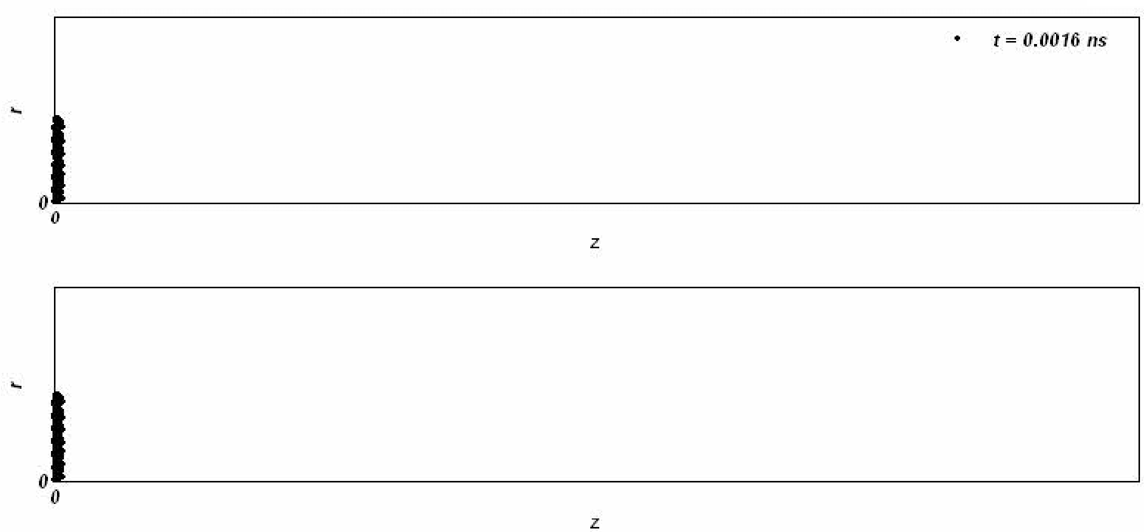
What is the Limiting Current?

Limiting Current – There is an upper bound on the amount of current that can be transported through a given geometry. This upper bound is referred to as the limiting current, and when exceeded, the system becomes unstable.





I_{below}

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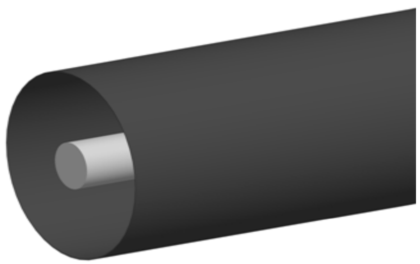
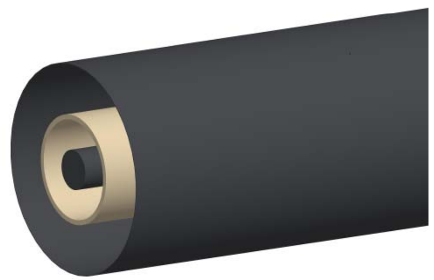
Outline

- Example of configurations where limiting current calculations are important
- Outline of methodology used to calculate limiting currents
  1. Electrostatic potential calculation
  2. Conservation of energy
  3. Optimization
- Example scenarios
  a. Simple case – All analytic approximation
  b. Medium case – Analytic calculation mixed with numerical calculation
  c. Complicated case – All numerical calculation

Example Configurations

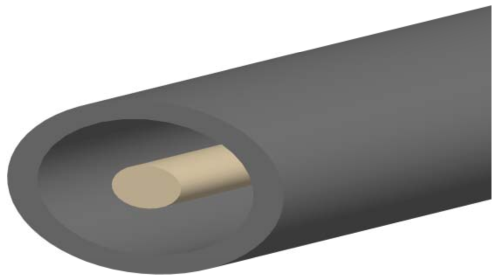
Simple cases. All analytic





More complicated. Analytic and numerical

Most complicated. All numerical

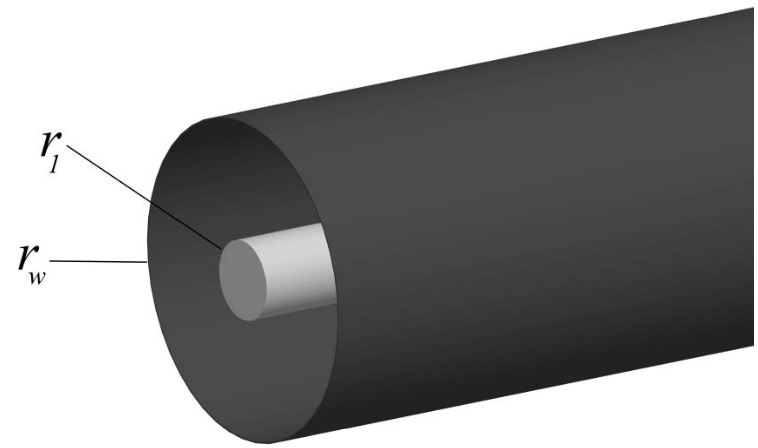


Simple all Analytic Case

Assume a UNIFORM current density and an initial energy of $E_b = \gamma_b mc^2$

$$\nabla^2 \phi = \frac{I}{\varepsilon_0 c A} \frac{\gamma}{\sqrt{\gamma^2 - 1}}$$

(1) $$\phi_0 = -\frac{I}{4\pi\varepsilon_0 c} \frac{\gamma}{\sqrt{\gamma^2 - 1}} g \qquad g = 1 + 2\ln\left(\frac{r_w}{r_1}\right)$$



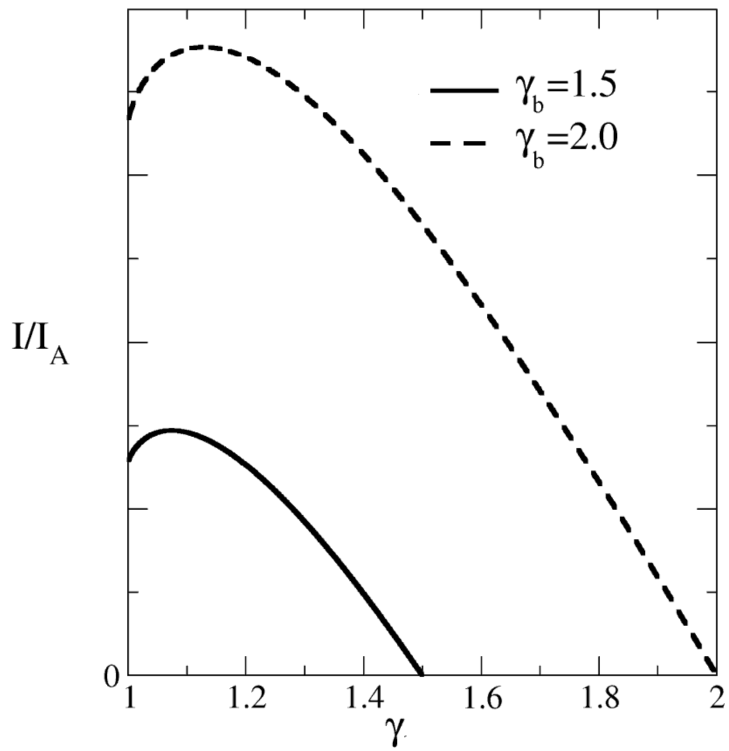
Substitute potential into relativistic conservation of energy equation

$$\gamma mc^2 = \gamma_b mc^2 + e\phi_0$$

Simple all Analytic Case

(2) $$\gamma = \gamma_b - \frac{I}{I_A}\frac{\gamma}{\sqrt{\gamma^2 - 1}}g$$

$I_A = 4\pi\varepsilon_0 mc^3 / e = 17.1kA$ for electrons

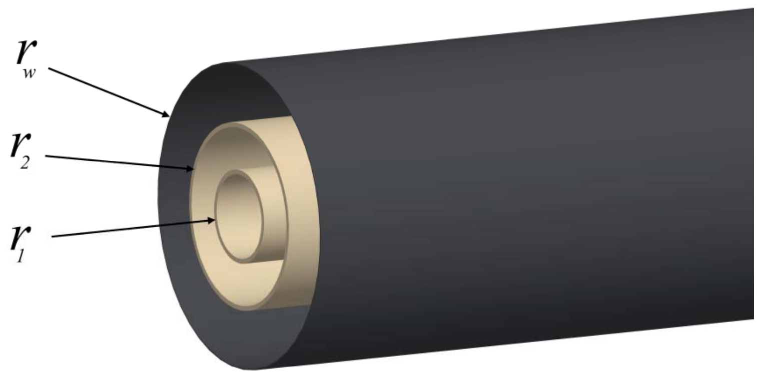


(3) *Optimize wrt γ in order to find maximum*

Occurs when $\gamma = \gamma_b^{1/3}$

$$\frac{I_{\lim}}{I_A} = \frac{\left(\gamma_b^{2/3} - 1\right)^{3/2}}{g}$$

More Complicated Case



$$\phi_n(r) = -\frac{I_n}{4\pi\varepsilon_0 c}\frac{\gamma_n}{\sqrt{\gamma_n^{\,2}-1}}\begin{cases} g_n & 0<r<r_n \\ 2\ln\left(r_w/r\right) & r_n<r<r_w \end{cases}$$

$$g_n = 2\ln\left(r_w/r_n\right)$$

$$\phi_T(r_1) = -\frac{I_1}{4\pi\varepsilon_0 c}\frac{\gamma_1}{\sqrt{\gamma_1^{\,2}-1}}g_1 - \frac{I_2}{4\pi\varepsilon_0 c}\frac{\gamma_2}{\sqrt{\gamma_2^{\,2}-1}}g_2 \qquad (1)$$

$$\phi_T(r_2) = -\frac{I_2}{4\pi\varepsilon_0 c}\frac{\gamma_2}{\sqrt{\gamma_2^{\,2}-1}}g_2 - \frac{I_1}{4\pi\varepsilon_0 c}\frac{\gamma_1}{\sqrt{\gamma_1^{\,2}-1}}g_2$$

$$\rightarrow \quad \gamma_n = \gamma_{b,n} + e\phi_T\left(r_n\right)/mc^2$$

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More Complicated Case

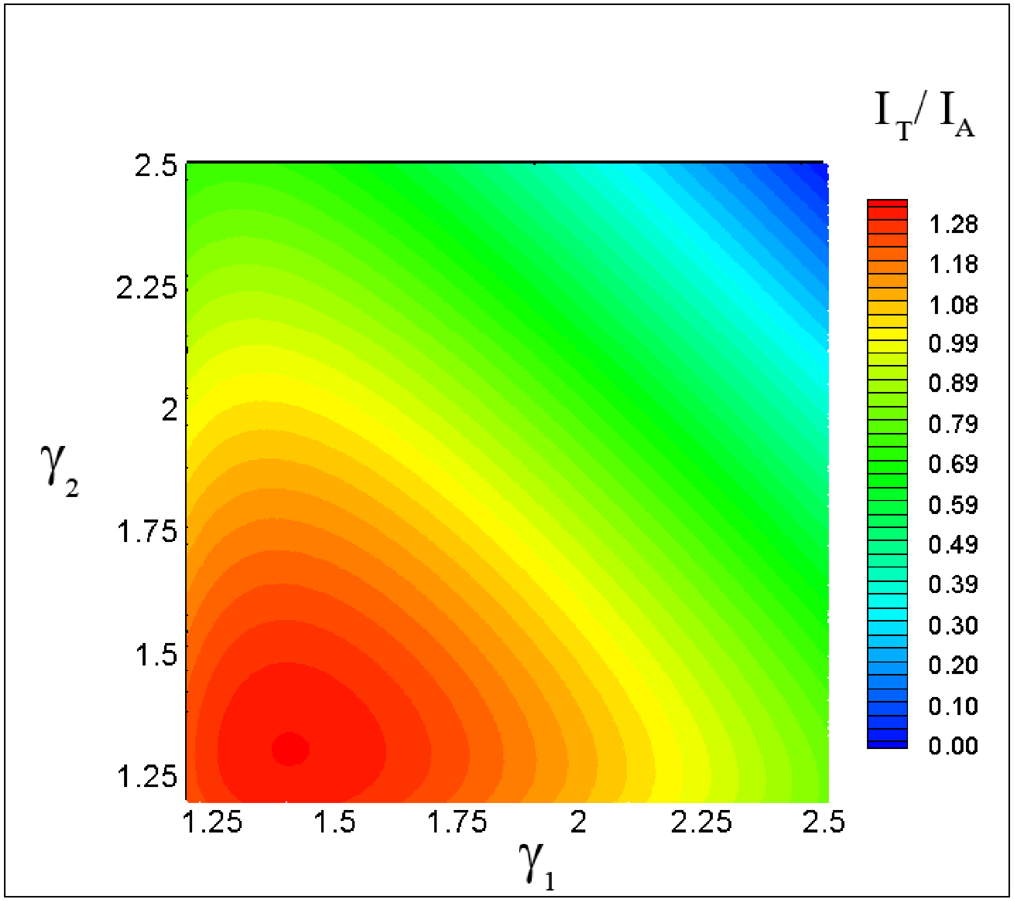
$$\gamma_1 = \gamma_{b,1} - \frac{I_1}{I_A}\frac{\gamma_1}{\sqrt{\gamma_1^2 - 1}} g_1 - \frac{I_2}{I_A}\frac{\gamma_2}{\sqrt{\gamma_2^2 - 1}} g_2 \qquad \gamma_2 = \gamma_{b,2} - \frac{I_2}{I_A}\frac{\gamma_2}{\sqrt{\gamma_2^2 - 1}} g_2 - \frac{I_1}{I_A}\frac{\gamma_1}{\sqrt{\gamma_1^2 - 1}} g_2$$

$$I_1 = f_1 I_T \qquad I_2 = f_2 I_T \qquad f_1 + f_2 = 1$$

$$\gamma_1 - \gamma_{b,1} + \gamma_2 - \gamma_{b,2} + (1 - f_2)\frac{I_T}{I_A}\frac{\gamma_1}{\sqrt{\gamma_1^2 - 1}}(g_1 + g_2) + 2 f_2 \frac{I_T}{I_A}\frac{\gamma_2}{\sqrt{\gamma_2^2 - 1}} g_2 = 0 \qquad (2)$$

- *We know $\gamma_{b,1}$, $\gamma_{b,2}$, f_2, g_1, and g_2*
- *We don't know γ_1 or γ_2 but we know they are bounded within $1 < \gamma_n < \gamma_{b,n}$*
- *Use Monte Carlo method to explore the γ_1 - γ_2 space and non-linear solver to find I_T at each point*

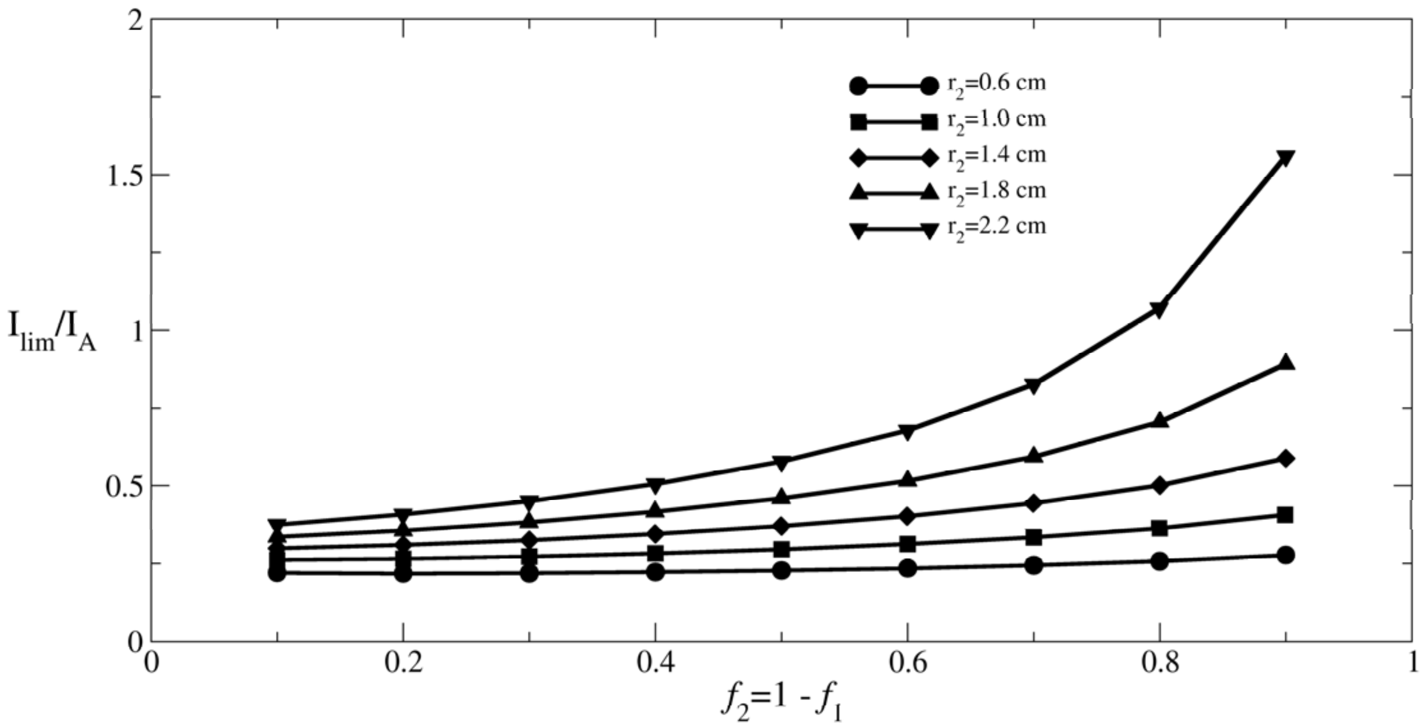
More Complicated Case



Total current versus energy of two beams. $r_1 = 1.5$ cm, $r_2 = 2.0$ cm, $r_w = 2.5$ cm, $f_2 = 0.5$, and $\gamma_{b,1} = \gamma_{b,2} = 2.5$

More Complicated Case

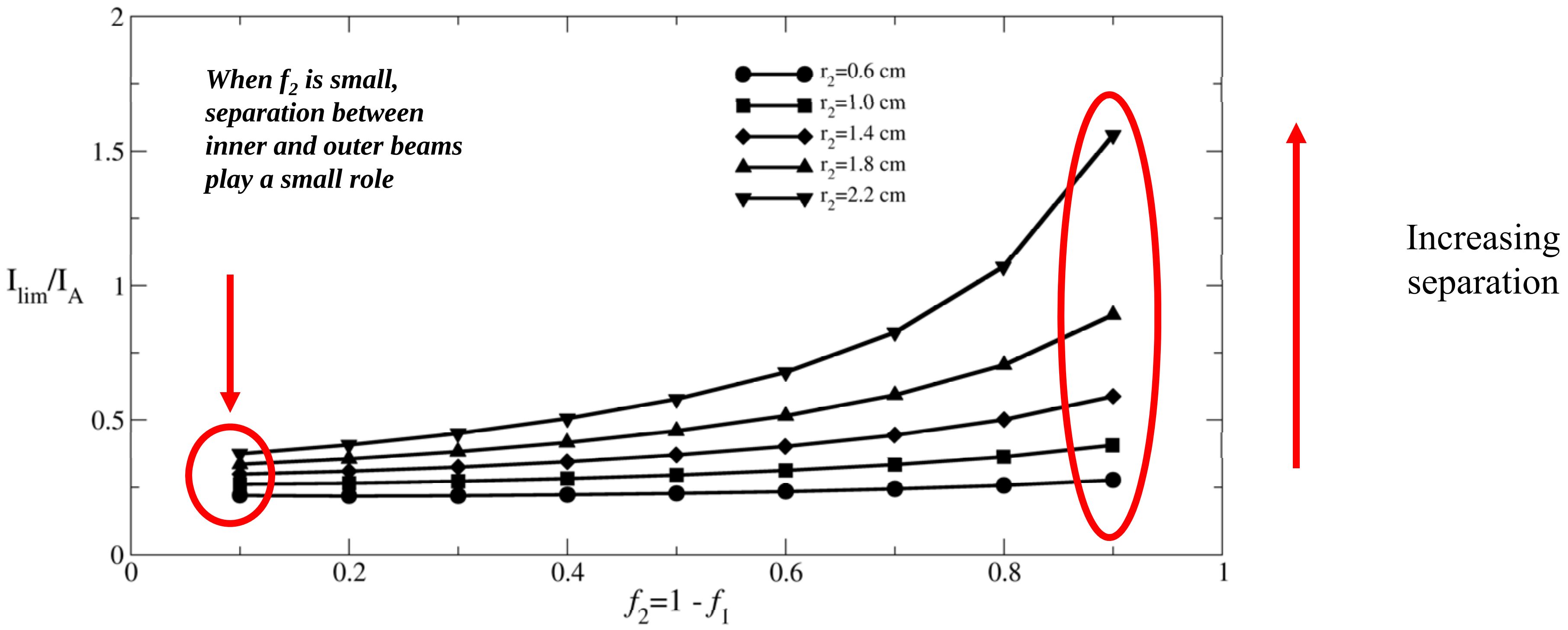
Normalized limiting currents as a function of $f_2=1-f_1$. Plot maintains r_1=0.2 cm. In both plots, $\gamma_{b,1} = \gamma_{b,2} = 2.5$, and r_w=2.5 cm



Increasing separation

More Complicated Case

Normalized limiting currents as a function of $f_2=1-f_1$. Plot maintains r_1=0.2 cm. In both plots, $\gamma_{b,1} = \gamma_{b,2} = 2.5$, and r_w=2.5 cm

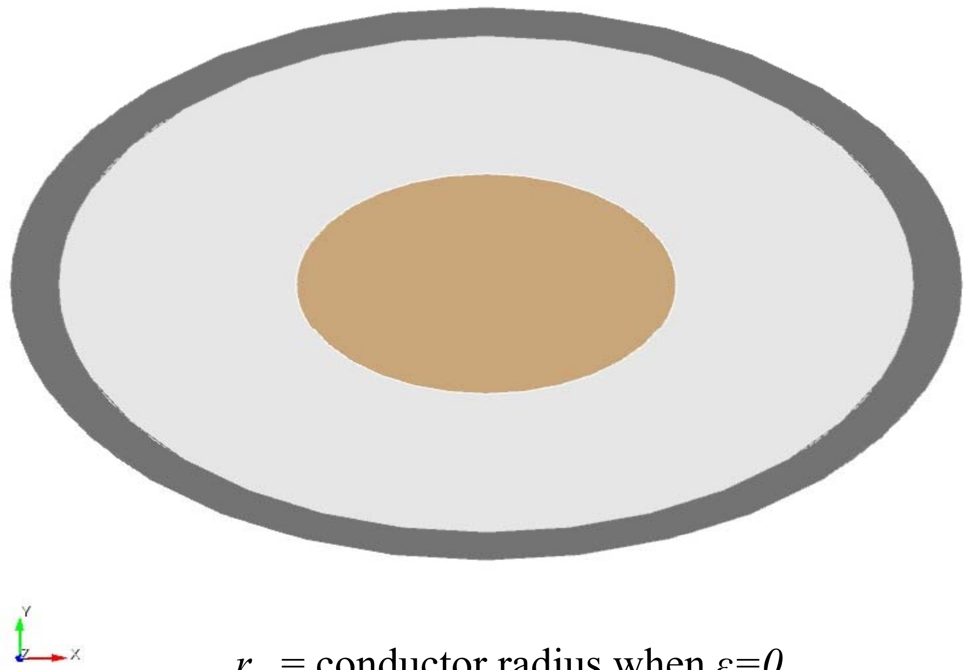


Most Complicated Case

Elliptic Beam

Becomes 2-d problem rather than 1-d like previous examples

Numerical solution to Poisson's Equation is required for the potential



r_w = conductor radius when $\varepsilon=0$

r_1 = beam radius when $\varepsilon=0$

Most Complicated Case

Finite Element Method (FEM) for PDE solution

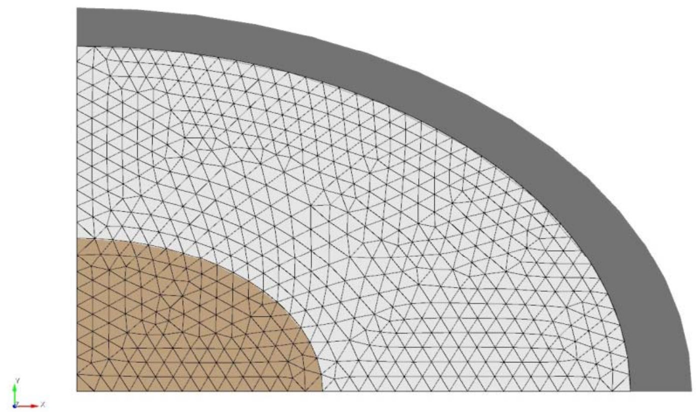
$$\nabla^2 \phi = \frac{4\pi}{A} \frac{I}{I_A} \frac{\gamma_b + \phi}{\sqrt{(\gamma_b + \phi)^2 - 1}}$$

→

Find solution that satisfies the following nonlinear system of equations:

$$\sum_j \left[K_{ij} \phi_j \right] - F_i(\phi_i) = 0$$

- Used Trilinos packages to numerically solve the PDE
  - Sundance: Finite Element Framework
  - NOX: Nonlinear system solver
  - AztecOO: Linear system solver



Quarter Symmetry Meshed Geometry

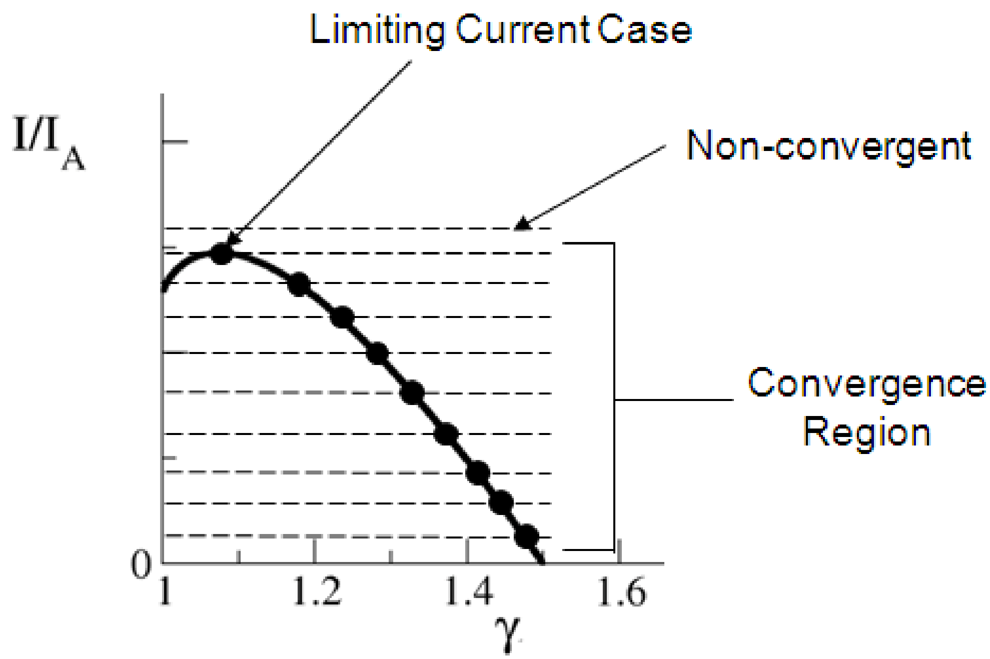
Most Complicated Case

$$\nabla^2 \phi = \frac{4\pi}{A} \frac{I}{I_A} \frac{\gamma_b + \phi}{\sqrt{(\gamma_b + \phi)^2 - 1}}$$

Search Algorithm

For a given beam geometry and γ_b

1. Select initial I/I_A
2. Solve PDE for potential
3. Does solution for potential converge?
   a. If converged, increment I/I_A and go to step 2
   b. Otherwise go to step 4
4. Last I/I_A with a valid solution is the limiting current



Most Complicated Case

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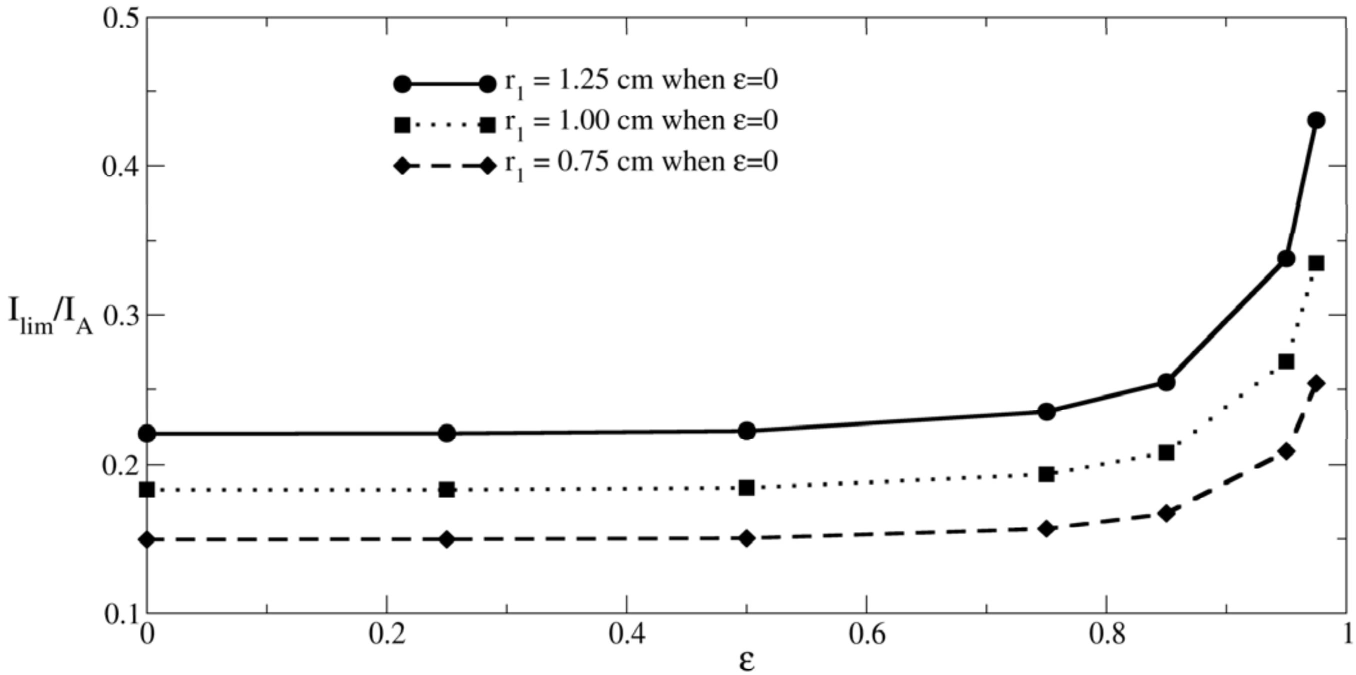

Limiting current versus the eccentricity for a solid beam when r_w = 2.5cm at $\varepsilon = 0$; Area of beam held constant

Closing Thoughts

- A following effort has begun which will replace the current methodology
  - Current method: Find the limiting current of a given geometry
  - Future method: Find the geometry which provides the greatest limiting current
- The new methodology is one of optimal shape design and parameter estimation
- This new method is in the class of PDE Constrained Optimization problems

Closing Thoughts

- PDE Constrained Optimization:
  - Allow the geometry (beam area and material properties) to vary and apply basic physical constraints to yield the following optimization problem:

$$\max \; I = f(J, A, \varepsilon)$$

$$\text{subject to } \nabla^2 \phi = \frac{4\pi}{A} \frac{I}{I_A} \frac{\gamma_b + \phi}{\sqrt{(\gamma_b + \phi)^2 - 1}}$$

$$0 \le \varepsilon < 1$$

  - where
    - f is the functional representing the limiting current in a specific configuration
    - J is the current density
    - A and ε are the beam area and eccentricity

Closing Thoughts

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Recent References

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